
Gauge Theory on the Fuzzy Sphere and Random Matrices

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1 Introduction

Gauge theories provide the best known description of the fundamental forces in nature. At very short distances however, physics is not known, and it seems unlikely that spacetime is a perfect continuum down to arbitrarily small scales. Indeed, physicists have started to learn in recent years how to formulate field theory on quantized, or noncommutative spaces.

However, most attempts to (second-) quantize these field theories using modifications of the conventional, perturbative methods have failed up to now. It seems therefore worthwhile to try to develop new techniques for their quantization, trying to take advantage of the peculiarities of the non-commutative case. There is indeed one striking feature of some non-commutative gauge theories: they can be formulated as *(multi-) matrix models*.

I will explain here the main ideas of [1], where such a matrix formulation of (pure) $U(n)$ gauge theory on the fuzzy sphere has been used to calculate its partition function in the commutative limit. This is done using matrix techniques which *cannot* be applied in the commutative case.

2 The model

Consider the matrix model for 3 hermitian $N \times N$ matrices B_i , with action

$$S(B) = \frac{2}{g^2 N} \text{Tr} \left((B_i B^i - \frac{N^2 - 1}{4})^2 + (B_i + i\epsilon_{ijk} B^j B^k)(B^i + i\epsilon^{irs} B_r B_s) \right) \quad (1)$$

where g is the coupling constant and N is a (large) integer. This action describes pure gauge theory on the fuzzy sphere S_N^2 , cp. [3, 4]. It is invariant under the $U(N)$ gauge symmetry acting as

$$B_i \rightarrow U^{-1} B_i U.$$

To see that this corresponds to the usual $U(1)$ local gauge symmetry in the classical limit, we first note that the absolute minimum (the “vacuum”) of the action is given by

$$B_i = \lambda_i = \pi_N(J_i)$$

up to gauge transformation, where π_N is the N -dimensional representation of $su(2)$ with generators J_i . Upon rescaling $\lambda_i = x_i \sqrt{\frac{N^2-1}{4}}$, one finds the generators x_i of the fuzzy sphere [3] which satisfy

$$\sum x_i x_i = 1, \quad [x_i, x_j] = i \sqrt{\frac{4}{N^2-1}} \epsilon_{ijk} x_k.$$

This means that the vacuum of this matrix model is the fuzzy sphere. We can now write any field (“covariant coordinate”) as

$$B_i = \lambda_i + \mathcal{A}_i. \quad (2)$$

Then

$$B^i + i\varepsilon^{ikl} B_k B_l = \frac{1}{2} \varepsilon^{ikl} F_{kl}, \quad F_{kl} := i[\lambda_k, \mathcal{A}_l] - i[\lambda_l, \mathcal{A}_k] + i[\mathcal{A}_k, \mathcal{A}_l] + \varepsilon_{klm} \mathcal{A}^m.$$

Notice that the kinetic terms in the field strength F_{kl} arise automatically due to the shift (2). The $U(N)$ gauge symmetry acts on \mathcal{A}_i as

$$\mathcal{A}_i \rightarrow U^{-1} \mathcal{A}_i U + U^{-1} [\lambda_i, U] \quad (3)$$

which for $U = \exp(ih(x))$ and $N \rightarrow \infty$ becomes the usual (abelian!) gauge transformation for a gauge field. One can furthermore show that the “radial” field

$$\varphi := \lambda^i \mathcal{A}_i$$

decouples in the large N limit, and $\frac{1}{N} Tr \rightarrow \int$. Hence the model reduces to the usual $U(1)$ Yang-Mills action

$$S = \frac{1}{g^2} \int F_{mn} F^{mn} \quad (4)$$

in the commutative (=large N) limit.

Monopoles.

One can find explicitly new, non-trivial solutions of this model using the ansatz [1]

$$B_i = \alpha_m \lambda_i^{(M)} \quad (5)$$

for suitable normalization constant α_m . Here $\lambda_i^{(M)} = \pi_M(J_i)$ is the generator of the M -dimensional irrep of $su(2)$, which can be embedded in the configuration space of $N \times N$ matrices if $m = N - M > 0$. It turns out that (5)

describes monopole solutions with monopole charge m , and the corresponding gauge potential can be calculated explicitly [1]. Notice that negative monopole charges $m < 0$ can also be obtained by admitting matrices B_i of size $N' \times N'$ with $N < N' \ll 2N$, while keeping the action (1) as it is.

Finally, it should be noted that the correct action $S = \frac{m^2}{2g^2}$ of the above monopoles is recovered only upon a slight modification of the action (1) as indicated in [1], which does not affect the classical limit. The reason is that the “empty” blocks in (5) if embedded in $N \times N$ matrices give a large contribution due to the first term in (1). This is certainly unphysical (it could be interpreted as action of a Dirac string), and can be avoided by the slightly modified action (78) in [1]. Then the energy of all monopoles is correctly reproduced in the commutative limit $N \rightarrow \infty$. All this extends immediately to the non-abelian case:

Non-abelian case

This model is readily extended to the nonabelian case by using matrices of size nN , i.e. $B_i = B_{i,\alpha} t^\alpha = \lambda_i t^0 + \mathcal{A}_{i,0} t^0 + \mathcal{A}_{i,a} t^a$ where t^a denote the Gell-Mann matrices of $su(n)$. The action then reduces to the usual $U(n)$ Yang-Mills action

$$S = \frac{1}{g^2} \int (F_{mn,0} F^{mn,0} + F_{mn,a} F^{mn,a}) \quad (6)$$

in the commutative limit. Again, all “instanton” sectors are recovered if one admits matrices of arbitrary size $M \approx nN$ for the above action.

3 Quantization

The quantization of $U(n)$ Yang-Mills gauge theory on the usual 2-sphere is well known, see e.g. [2, 5]. In particular, the partition function and correlation functions of Wilson loops have been calculated. Our goal is to calculate the partition function for the YM action (1) on the fuzzy sphere, taking advantage of the formulation as matrix model. This can be achieved by collecting the 3 matrices B_i into a single $2M \times 2M$ matrix

$$C = C_0 + B_i \sigma^i \quad (7)$$

The main observation is that the above action (1) can be rewritten simply as

$$S(B) = Tr V(C)$$

imposing the constraint $C_0 = \frac{1}{2}$, for the potential

$$V(C) = \frac{1}{g^2 N} (C^2 - (\frac{N}{2})^2)^2$$

Then we proceed as

$$\begin{aligned}
Z &= \int dB_i \exp(-S(B)) \\
&= \int dC \delta(C_0 - \frac{1}{2}) \exp(-\text{Tr}V(C)) \\
&= \int d\Lambda_i \Delta^2(\Lambda_i) \exp(-\text{Tr}V(\Lambda)) \int dU \delta((U^{-1}\Lambda U)_0 - \frac{1}{2})
\end{aligned}$$

where dU is the integral over $2M \times 2M$ unitary matrices, $C = U^{-1}\Lambda U$, and $\Delta(\Lambda_i)$ is the Vandermonde-determinant of the eigenvalues Λ_i . Here $\delta(C_0 - \frac{1}{2})$ is a product over M^2 delta functions, which can be calculated by introducing $J = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} = K \sigma^0$ where K is a $N \times N$ matrix. Then

$$\delta((U^{-1}CU)_0 - \frac{1}{2}) = \int dK \exp(i\text{Tr}(U^{-1}(C - \frac{1}{2})UJ)).$$

By gauge invariance, the r.h.s. depends only on the eigenvalues Λ_i of C . Hence

$$\begin{aligned}
Z &= \int dK \int d\Lambda_i \Delta^2(\Lambda_i) \exp(-\text{Tr}V(\Lambda)) \int dU \exp(i\text{Tr}(U^{-1}\Lambda UJ - \frac{1}{2}J)) \\
&= \int dK Z[J] e^{-\frac{i}{2}\text{Tr}J}
\end{aligned} \tag{8}$$

where

$$Z[J] := \int dC \exp(-\text{Tr}V(C) + i\text{Tr}(CJ)) \tag{9}$$

depends only on the eigenvalues J_i of J . Diagonalizing $K = V^{-1}kV$, we get

$$Z = \int dk_i \Delta^2(k) \int d\Lambda_i \Delta^2(\Lambda_i) \exp(-\text{Tr}V(\Lambda)) \int dU \exp(i\text{Tr}(U^{-1}(\Lambda - \frac{1}{2})UJ))$$

where $\int dV$ was absorbed in $\int dU$. The integral over $\int dU$ can now be done using the Itzykson-Zuber-Harish-Chandra formula [6]

$$\int dU \exp(i\text{Tr}(U^{-1}CUJ)) = \text{const} \frac{\det(e^{i\Lambda_i J_j})}{\Delta(\Lambda_i)\Delta(J_j)},$$

which also depends only on the eigenvalues of J and C .

In this step the number of integrals is reduced from N^2 to $2N$. This basically means that the integral over fields on S_N^2 is reduced to the integral over functions in one variable. This is a huge step, just like in the usual matrix models. The constraint however forces us to evaluate in addition the integral over k_i , which is quite complicated due to the rapid oscillations in $\det(e^{i\Lambda_i J_j})$; note that $\Lambda_i \approx \pm \frac{N}{2}$. Nevertheless, the integrals can be evaluated for large N [1], with the result

$$Z_m = \sum_{m_1 + \dots + m_n = m} \int_{-\infty}^{\infty} d\kappa_1 \dots d\kappa_n \Delta^2(\kappa) e^{i\kappa_i m_i} \exp(-\frac{g^2}{2} \sum \kappa_i^2).$$

Here we consider matrices of size $M = nN - m$, which corresponds to the monopole sectors with total $U(1)$ charge $m = m_1 + \dots + m_n$. This can be rewritten in the “localized” form as a weighted sum of saddle-point contributions, as advocated by Witten [5]:

$$Z_m = \sum_{m_1 + \dots + m_n = m} P(m_i, g) \exp\left(-\frac{1}{2g^2} \sum m_i^2\right)$$

where $P(m_i, g)$ is a totally symmetric polynomial in the m_i which can be given explicitly. In order to include all monopole configurations, we should simply sum over matrices of different sizes $M = nN - m$, for the same action given by $V(C)$. One can indeed find corresponding saddle-points of the action (1) which have the form [1]

$$A_i = \begin{pmatrix} m_1 A_i & 0 & \dots & 0 \\ 0 & m_2 A_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m_n A_i \end{pmatrix}$$

where

$$\mathbf{A}_i = \mathbf{r} \times \mathcal{A} \approx \frac{m}{2} \frac{1}{1 + x_3} \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}$$

becomes the usual monopole field for large N , and action becomes

$$S(C^{(m_1, \dots, m_n)}) = \frac{1}{2g^2} \sum_i m_i^2$$

for large N . This is a standard result on the classical sphere.

Hence the full partition function is obtained by summing¹ over all Z_m ,

$$Z = \sum_m Z_m = \sum_{m_1, \dots, m_n = -\infty}^{\infty} \int d\kappa_i \Delta^2(\kappa) e^{i\kappa_i m_i} \exp\left(-\frac{g^2}{2} \sum \kappa_i^2\right).$$

Using a Poisson resummation, this can be rewritten in the form

$$Z = \sum_{p_1, \dots, p_n \in \mathbb{Z}} \Delta^2(p) \exp\left(-2\pi^2 g^2 \sum_i p_i^2\right)$$

or equivalently

$$Z = \sum_R (d_R)^2 \exp(-4\pi^2 g^2 C_{2R}).$$

¹ the relative weights of Z_m for different m is not determined here. However, it could be calculated in principle

Here the sum is over all representations of $U(n)$, d_R is the dimension of the representation and C_{2R} the quadratic casimir. This form was found in [2] for the partition function of a $U(n)$ Yang-Mills theory on the ordinary 2-sphere.

We see that the limit $N \rightarrow \infty$ of the partition function for $U(n)$ YM on the fuzzy sphere is well-defined, and reproduces the result for YM on the classical sphere. This strongly suggests that the same holds for the full YM theory on the fuzzy sphere, and that there is nothing like UV/IR mixing for pure gauge theory on S_N^2 . This is unlike the case of a scalar field, which exhibits a “non-commutative anomaly” [7] related to UV/IR mixing.

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